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INCOMPLETENESS AND LOGIC

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Introduction

In this book I will speak about unprovable truths within mathematics, namely truths that principles of mathematics cannot prove. I will focus my attention on Gödel's sentences and the Continuum Hypothesis. You have to pay attention to the fact that the Continuum Hypothesis was considered by David Hilbert as the first mathematical problem to be solved. This book is constituted by two parts (e.g. two main chapters). In the first part, I will address mathematical issues related to arithmetic. In this part i will explain Gödel's theorems. This part must be considered as a sort of introduction to the part about the continuum hypothesis. This book might be considered a kind of travel where we can see how the phenomenon of incompleteness (e.g. unprovable truths) arises within mathematics.

In order to start to speak about Gödel's theorems we have to depart from Liar paradox (ancient greek paradox). To examine this paradox we must analyze the following sentence: *this sentence is false*. This sentence is saying of itself that it is false. It is a self-referential sentence. This sentence is a paradox since it is at the same time true and false. If we reason about this paradox we can say that if it is false, since it is saying of itself that is false, then it is true and if it is true, since it is saying of itself that is false, then it is false. Therefore this sentence is at the same time true and false. We cannot establish whether it is true or false. We cannot escape from paradoxes. Another paradox, that caused many problems to logicians, was Russelll paradox (1906). If we take the class of all classes that do not belong to themselves, we can ask ourselves: Does the Russellian Class belong to itself? so, if it belongs to itself, since it the Class of all classes that do not belong to themselves, it does not belong to itself and if it does not belong to itself, since it is the Class of all classes that do not belong to themselves, it belongs to itself. This is another paradox which threatens the foundation of mathematics. Russell paradox forces us to abandon the idea that every property can define a set (e.g. naive Fregean abstraction principle). Both paradoxes do not have an immediate solution. To avoid them we have to use meta-languages or we have to restrict the concepts used within set theory. The Liar paradox inspired Kurt Gödel to construct Gödel's sentence, a truth that mathematical principles cannot prove. Kurt Gödel was able to construct an arithmetical sentence that is saying: *I am unprovable*. Even if this sentence is self-referential and resembles Liar paradox, it is not paradoxical, but it is a perfect arithmetical sentence. (i.e. Non-standard view as we will see). After the procedure of decoding (e.g. I will explain later) we discover that this sentence is saying: *I am unprovable*. Around 1930 many mathematicians were believing that the theory of arithmetic (e.g. the theory of numbers 0, 1, 2, 3) was complete. What does it mean to be complete? it means that the principles of arithmetic could prove all truths. If we have a truth, principles of arithmetic could prove it by deducing it from these basic, evident principles. An axiomatic system is a set of these basic, evident principles from which we can deduce all truths by adopting truth-preserving rules. In arithmetic we have Peano axiomatic system. Peano axiomatic system is a set of seven, evident, basic principles from which we can deduce truths regarding finite numbers (e.g. arithmetic). So, at this point we can ask ourselves: is Peano axiomatic system complete? can we prove all truths regarding arithmetic from Peano's principles? Since Presburger's arithmetic was complete and Skolem's srithmetic was complete, many mathematicians were believing that also Peano's arithmetic was complete. We could derive all truths regarding finite numbers from Peano's principles by following truth-preserving rules. Unfortunately Kurt Gödel proved that Peano axiomatic system is incomplete. There are truths that Peano principles cannot prove. Kurt Gödel in 1931 showed his results that are called incompleteness theorems. There are truths that no axiomatic system sufficiently strong can prove. Therefore there are unprovable truths. So, now we can see how Kurt Gödel was able to construct this sentence true but unprovable. By using Gödel numerical coding, Kurt Gödel was able to construct a numerical sentence that when decoded is saying of itself: *I am umprovable*. This sentence is true since it is unprovable (i.e. Standard view or naive view). By adopting Gödel coding syntactic properties become numerical properties. Being an Axiom, being a sentence, being a sentence derived by modus ponens become simple numbers. Thanks to Gödel coding, we can define a numerical property $\mathbf{Bew}(m, n)$ which holds when m is a code number of a proof of the sentence with code number n. The property of being an axiom, a derivation, a mathematical proof, thanks to Gödel coding, become numerical properties. The self referential Gödel sentence, which says of itself to be unprovable, becomes a Gödel number and it is self referential only when we translate back from Gödel numbering. We have a numerical language which, unlike natural language, is precise and it does not have problem of denotation. Now we have to see whether Peano' s axiomatic system is sufficiently strong. Peano's axiomatic system is sufficiently strong since it can capture all primitive recursive functions. Primitive recursive functions are computable by definition (as we will see later). We can compute primitive recursive functions!. Kurt Gödel was able to show after constructing a chain of primitive recursive functions that the relation **Bew**(m, n), which holds when m is a code number of a proof of the sentence with code number n, was primitive recursive. Thus, Peano axiomatic system could capture it and was sufficiently strong. So, Kurt Gödel could use it in order to construct the numerical sentence that, when decoded, says of itself: *I am unprovable*.

Gödel's arithmetical undecidable statements are not absolutely undecidable. There is a sense, however, in which we can consider them benign since to the extent that we are justified in accepting PA we are justified in accepting Con(PA) and so we can expand the axiom system to solve incompleteness. There are three ways to capture undecided sentences. As we will see in section 10, Turing's approach (1936) of transfinite progression is an example. Secondly we can consider Feferman's approach (1991). When we accept PA, we accept also any meaningful predicate on natural numbers. So we are justified in accepting the system obtained by expanding the language to include the truth predicate and allowing the truth predicate to figure in the induction scheme. The expanded system can prove Con(PA). The procedure can be iterated into the transfinite and it gives origin to a system known as predicative analysis. Thirdly we have the most natural approach since it involves moving to the system of next higher type, allowing variables that range over subsets of natural numbers (i.e. real numbers). This system, called second order arithmetic (i.e. PA_2), proves Con(PA). Kurt Gödel, when was conceiving a possible solution to arithmetical undecided sentences was keen on this third approach.

Furthermore in the second part of this book I will address the notion of absolute provability that implies the general completeness theorem advocated by Gödel. The Bohemian logician writes:

> It is not impossible that for such a concept of demonstrability some completeness theorem would hold which would say that every proposition expressible in set theory is decidable from the present axioms plus some true assertions about the largeness of the universe of all sets. [Kurt Gödel (1946) in [Kurt Gödel 1990 p. 151]]

Kurt Gödel in this passage is expressing the thesis that every problem within set theory can be decided. We encounter the concept of absolute demonstrability. This quotation is similar to the Hilbert's mantra, namely *No Ignorabimus*, or Leibniz's mantra, *Calculemus*. This belief suggests that we do not have absolute undecidable problems but every mathematical proposition can be settled. The Continuum Hypothesis can be decided if we discover a general completeness theorem. Woodin's two main projects [Woodin 2010] [Woodin 2010b] belong to this conception that sees incompleteness as residual, not central to the mathematical practice. Thus, if the incompleteness phenomenon is residual or we might say that is an epiphenomenon within mathematics, every set-theoretic problem can be settled including the Continuum Hypothesis. So we have to introduce new principles to obtain a general completeness theorem. Maybe a solution to open problems does not come from mathematics but from philosophy, namely a conceptual analysis of the concept of set. Here we have Gödel's quotation expressing this idea:

> This scarcity of results, even to the most fundamental questions in this field, may be due to some extent to purely mathematical difficulties; it seems, however [. . .] that there are also deeper reasons behind it and that a complete solution of these problems can be obtained only by a more profound analysis (than mathematics is accustomed to give) of the meanings of the terms occurring in them (such as set, one-to-one correspondence, etc) and of the axioms underlying their use.[Kurt Gödel 1947 p. 517]

So if a solution comes from philosophy, we have to analyze the concept of set. In the second part of this book we will analyze two conjectures that imply Gödel's general completeness theorem, namely the V=L Hypothesis and the Ultimate L conjecture. The V=L Hypothesis is the following statement:

Definition 1 *(V=L Hypothesis) ZFC + (true) reflection principles (Koellner's program) + Jensen's covering theorem + Hamkins' Maximality principle + Multiverse principle + Putnam's closure condition + V=L ⟶ Gödel's general completeness theorem.*

Chapter I

The Dream of completeness

Preliminaries to this chapter

In this chapter I will discuss formally (mathematical language) the phenomenon of incompleteness in arithmetic. I will discuss how the phenomenon of incompleteness, discovered by Gödel, appears in first-order arithmetic. I will examine different axioms that were assumed by mathematicians to settle undecided questions.I will introduce Gödel's incompleteness theorems. Gödel's sentences are unprovable truths of first-order arithmetic. Then I will explain Turing's completeness result about transfinite progressions. Turing, by going into the transfinite, attempted to settle first-order arithmetical sentences including Gödel's sentences. Unfortunately, Turing's attempt was doomed to fail because of a problem connected with ordinal notation, as we will see. Sometimes mathematicians assert that Gödel's sentences are not mathematically interesting. Therefore I will introduce Goodstein's theorem and the Finite extension of Ramsey theorem which are considered mathematically interesting and were shown to be undecidable within Peano arithmetic. Then I will discuss Isaacson's conjecture and by assuming the non-standard view about Gödel's sentence, I will argue that this conjecture might be false. I will conclude this chapter by introducing Chaitin's magical Ω numbers and I will discuss randomness. I will show by following Chaitin's results that randomness implies incompleteness.

1.1. Gödel's theorems

1.1.1. *Prerequisites to this section*

The language of arithmetic consists of first-order logic apparatus and the following symbols: 0-ary function symbol (costant) 0, unuary function symbol S (the successor function), two binary function symbols $+, \times$, two binary relation symbols $=, <$ and for each n, infinitely many n-ary predicate symbols X_n. Now we can introduce Levy's hierarchy. A formula ϕ is Σ_0 or Π_0 (Δ_0) if and only if it does not contain unbounded quantificators. For

$n \geq 1$, by recursion, we assert that ϕ is Σ_n if and only if has the form $\exists \tilde{x} \psi(\tilde{x})$ where $\psi(\tilde{x})$ is Π_{n-1}. and that ϕ is Π_n if and only if it has the following form $\forall \tilde{x} \psi(\tilde{x})$ where $\psi(\tilde{x})$ is Σ_{n-1}. Therefore, when we assert that a formula is Σ_n, we want to say, first of all, that it consists of a Δ_0 formula which has n blocks of existential quantificators in front. Secondly, this formula starts with a block of existential quantificators. Thirdly, this formula is characterized by an alternation of blocks of universal quantificators and blocks of existential quantificators. A formula is Δ_1 if it is equivalent to both a Σ_1 and a Π_1 formula. Usually, we will use also superscripts that point out to the order of formulas. For example a Π^0_1 formula starts with an unbounded block of universal quantificators and it is a first-order formula. Let $n > 0$ be a natural number and let us consider the nth order predicate calculus. There are variables of orders 1, 2, ...,n and the quantifiers are applied to variables of all orders. An nth order formula contains, in addition to first-order symbols and higher order quantifiers, predicates $X(z)$ where X and z are variables of order $x + 1$ and x respectively (for any $x < n$). Satisfaction for an nth order formula in a model $M = (A, P, \ldots, f, \ldots, c, \ldots,)$ is defined as follows: variables of first-order are interpreted as elements of the set A, variables of second-order as elements of P(A) (as subsets of A), etc; variables of order n are interpreted as elements of $P^{n-1}(A)$. The predicate $X(z)$ is interpreted as $z \in X$. A Π^n_m formula is a formula of order $n + 1$ of the form $\forall X \exists Y \ldots \psi$ (m quantifiers) where X, Y, are $(n + 1)th$ order variables and ψ is such that all quantified variables are of order at most n. Similarly, a Σ^n_m formula is the same but with $\exists$ and $\forall$ interchanged. See [Jech 2006]

1.1.2. *Preliminaries to this section*

In the first two sections we will become aware that the phenomenon of incompleteness appears naturally in first order arithmetic. To escape from incompleteness, we have to make very strong assumptions. In section 2 I will present some notions of computability. I will define the notions of primitive recursive functions and partial recursive functions. Then, I will explain Church's thesis and I will discuss it philosophically in connection with the consistency of ZFC and Intuitionism. Finally, I will introduce Turing's Universe and Turing's degrees of computability. Gödel's first incompleteness theorem establishes that there is a missmatch between truth and theoremhood within PA. This section aims at showing what is the distance between truth and theoremhood within PA in terms of Turing's degrees of computability. In this section, I will introduce also some notions related to intuitionism. In fact, I will argue that Church's thesis can be considered as potentially true but it cannot be seen as an atemporal truth. In section 4 I will discuss Gödel's incompleteness theorems. I will show

how it is possible to construct a Gödel's sentence. In this section we will discuss how the phenomenon of incompleteness was discovered by Gödel in 1931. In section 5 we discuss statements unprovable within PA mathematically interesting (Goodstein's theorem and the extended finite Ramsay theorem).Sometimes mathematicians say that Gödel's sentences are not mathematically interesting. So, I want to consider Goodstein's theorem and an extension of the finite Ramsey theorem, two arithmetical statement which PA cannot prove. So, we can say that the phenomenon of incompleteness is an essential feature of first-order arithmetic. I will conclude this part by examining Isaacson's conjecture and by assuming the non-standard view I will assert that Gödel's sentence is perfectly arithmetical sentence and so we might disprove Isaacson's conjecture.

1.2. Brief introduction to unprovable truths

I entitled this chapter the dream of completeness because at the beginning of the last century many mathematicians believed that all mathematical truths could be proved. The axiomatic systems, such as Peano arithmetic and Zermelo-Frankel axiomatic set theory, were considered to be complete. We could prove all truths by deducing them from the axioms. A theory is complete if for every formula, the theory can prove the formula itself or its negation. Unfortunately, in 1930, Kurt Gödel proved that no consistent axiomatic theory that is sufficiently strong is complete. There are truths that cannot be proved. The day after Gödel communicated his famous result to a philosophical meeting in Könisberg, in September 1930, David Hilbert could be found in another part of the same city delivering the opening address to the Society of German Scientists and Physicians, famously declaring:

> For the mathematician there is no Ignorabimus, and, in my opinion, not at all for natural science either. . . The true reason why (no one) has succeeded in finding an unsolvable problem is, in my opininion, that there is no unsolvable problem. In contrast to the foolish Ignorabimus, our credo avers: We must know, We shall know. [Cooper 2007 p. 5]

For the first incompleteness theorem there is a sentence (Gödel sentence) that is true but unprovable within Peano axiomatic number system. Gödel sentence says that *I am unprovable* and it is true because it is unprovable. At the first look, it can seem a self-referential sentence which is similar to the liar paradox, but it is not the case. In fact, for Gödel's coding (as we will see later), Gödel sentence is an arithmetical sentence expressed in the language of arithmetic. Only at the moment that we decode the sentence we discover that this sentence says of itself to be unprovable. So Peano axiomatic system, which aims at pinning down the structure of natural numbers is incomplete. There are truths that cannot be proved.

Let us introduce the axioms of Peano's first-order axiomatic system (PA).

The language of PA is a first-order language whose non-logical vocabulary includes the constant 0 (zero), the one-place function S (the successor function) and the two-place functions + (addition) and × (multiplication). The axioms are the following:

a) $\forall x(0 \neq Sx)$
b) $\forall x \forall y(Sx = Sy \longrightarrow x = y)$
c) $\forall x(x + 0 = x)$
d) $\forall x \forall y(x + Sy = S(x + y))$
e) $\forall x(x \times 0 = 0)$
f) $\forall x \forall y(x \times Sy = (x \times y) + x)$
g) (Induction schema) $\phi(0) \wedge \forall x(\phi(x) \longrightarrow \phi(S(x)) \longrightarrow \forall x \phi(x)$, for every formula.

The most problematic axiom is the Induction schema, since by assuming this axiom, we are refering to numerical properties. Thus, ideally we should be able to quantify over numerical properties (sets). So we should adopt a second-order version of it. But in first-order axiomatic system, quantifiers range over the domain of numbers, so we are forced to adopt first-order language. The solution is represented by the fact that we use a schema. Thus, any first-order formula expressing a property which fits the template is an induction axiom.

An important subsystem of Peano axiomatic system is Robinson's arithmetic, (**Q**), which has the following axioms:

a) $\forall x(0 \neq Sx)$
b) $\forall x \forall y(Sx = Sy \longrightarrow x = y)$
c) $\forall x(x \neq 0 \longrightarrow \exists y(x = Sy))$
d) $\forall x(x + 0 = x)$
e) $\forall x \forall y(x + Sy = S(x + y))$
f) $\forall x(x \times 0 = 0)$
g) $\forall x \forall y(x \times Sy = (x \times y) + x)$

Q is a sound theory, its axioms are all true in the standard model of arithmetic and its logic is truth-preserving. But, **Q** is incomplete. There are very simple true quantified sentences that **Q** cannot prove. It cannot prove universal generalizations. Since **Q** lacks the induction schema, it cannot handle all quantified sentences. However, although Robinson's arithmetic is a weak theory, it is very interesting. In fact, **Q** is sufficiently strong. This weak subsystem of Peano's arithmetic is Σ_1-complete. It can prove all true Σ_1 sentences. Furthermore, all primitive recursive functions can be expressed by a Σ_1 formula in Q[1] sentences. Therefore, **Q** can represent all primitive recursive functions including the demonstrability predicate, fundamental in the construction of the undecidable Gödel sentence. Suppose a theory of arithmetic is formally axiomatized, consistent and can prove everything that **Q** can prove (a very weak requirement). Then this theory will be sufficiently strong and so will be incomplete since it will be possible within this theory to construct Gödel's undecidable sentence.

The first incompleteness theorem undermines *Principia Mathematica*'s logicism[2]. However in 1931, the logicist project was over. Instead, the dominant project was Hilbert's program which aimed at showing that infinitary mathematics was not contradictory. Hilbert was thinking that we should divide mathematics into a core of uncontentious real mathematics and a superstructure of ideal mathematics. Propositions of real mathematics are simply true or false. Four plus two is six and two plus one is three. We could say according to the simplicity of the statements [Smith 2007 p. 53] that Π_1-statements of arithmetic belong to Hilbert's uncontentious real mathematics. We will discover later that many Π_1-statement are unprovable, such as Gödel sentence, the consistency statement (Gödel second incompleteness theorem) and Goldbach's conjecture whereas other Π_1 statements are provable such as the Last theorem of Fermat. By contrast, ideal mathematics shouldn't be thought of as having representational content and its sentences aren't strictly-speaking true or false. In pursuing this idea, Hilbert took a very restricted view of real mathematics. Influenced by Kant, Hilbert thought that the most certain of arithmetic was grounded on intuition, which enabled us to understand finite sequences of numbers and results when we manipulated them. Hilbert's view is characterised by two components, namely strict finitism and a formalistic approach towards mathematics. For the German mathematician mathematics is represented by finite strings of symbols that we manipulate. Maybe we can identify what Hilbert was thinking by using the term *real core mathematics*, with the theory PRA, namely first-order arithmetic plus primitive recursive functions. In fact from one side PRA is a theory about arithmetic and from the other side it is strong enough to capture all primitive recursive functions. So according to Hilbert's view, we must distinguish real core mathematics

1. In the language of arithmetic Δ_0 formulas are bounded formulas built up using identity, the less-then-or-equal relation, propositional connectives and bounded quantifiers. Σ_1 formulas are unbounded existential quantifications of Δ_0 formulas and Π_1 are universal unbounded quantifications of Δ_0 formulas.

2. We mean by Logicism a theory which implies that all arithmetical truths can be derived from basic, self-evident, logical truths. This theory aims at constructing mathematics upon logic.

from its ideal superstructure (such as set theory). Then you want to know which bits of ideal mathematics are safe to use, are real-sound, namely what ideal mathematics proves is true. For this one has to find which parts of ideal mathematics can be proved finitistically consistent. A corollary of the first Gödel incompleteness theorem was the second Gödel incompleteness theorem which states: no consistent sufficiently strong theory can prove its own consistency. Robinson's arithmetic (**Q**) and Peano arithmetic (PA) cannot have a proof of their own consistency. So no modest formal arithmetic can establish the consistency of a fancy ideal theory. So we cannot have consistency proofs for branches of ideal mathematics. Therefore, Hilbert's project of trying to establish the real soundness of ideal mathematics by giving consistency proofs using real and contentual mathematics was demolished by Gödel's second incompleteness theorem.

Returning to Gödel's first incompleteness theorem, we have that Gödel sentence is unprovable or undecidable. We can also say that it is incomputable. We use the term computable for functions, namely computable by a Turing machine or by recursion, when the informal instructions of an algorithm are made formal. Using the term computable truth means that we can give a proof of that truth (tree proof or linear sequence proof). At this point, we have to clarify the concept of truth in mathematics: why a mathematical sentence is true? We could answer that a mathematical sentence is true because it is proved within the axiomatic system such as PA, or outside the system, or because there is an independent mathematical reality which makes the sentence true. However, mathematical truth is a definite and precise mathematical property that we express by inductive definitions. Alfred Tarski introduced inductive definitions of truth which made the notion of truth a precise mathematical property. Gödel proved his two incompleteness theorems by looking outside the formal system[3] and when we come across Gödel sentence, we discover that it is true because it is unprovable. So there is a strong link between truth and provability in mathematics, but thanks to Gödel's theorem we can say that there is a miss-match between truths and proofs. I entitled this section *the dream of completeness* yet around 1929 many mathematicians were believing that it would have been possible that Peano axiomatic system was complete. In fact in 1929 Mojźesz Presburger proved that the theory P (PA Peano arithmetic minus multiplication) was complete. In the same year, Thoralf Skolem proved that a theory with multiplication, but lacking addition, was complete. Therefore, many mathematicians were hoping that also Peano arithmetic was complete. It is interesting to know that Presburger used in his proof a model-theoretic procedure (quantifier elimination) which also

3. In 1938 Hilbert and Bernays gave a formal proof of Gödel's theorems within the system.

Alfred Tarski later adopted to show that the theory of real closed fields is complete. Therefore in 1929 many mathematicians were thinking that also Peano arithmetic PA would be a complete theory. In fact, even Gödel attempted to prove the completeness of Peano arithmetic. But if arithmetic with multiplication minus addition, and arithmetic with addition minus multiplication, are complete theories we should ask ourselves why when we put together these two operations we have the phenomenon of incompleteness. The reason is that thanks to addition and multiplication we can construct a chain of primitive recursive functions and we can show at the end that the predicate of demonstrability **Bew** is primitive recursive. Since in Peano arithmetic all primitive recursive functions are representable, also the predicate of demonstrability is representable and so we can construct Gödel's sentence which says of itself to be unprovable. Sometimes mathematicians assert that Gödel's sentences are not mathematically interesting.

1.3. Turing's universe

At this point, before constructing Gödel's sentence, I want to speak a little about computability. This section aims at showing what is the distance between truth and theoremhood within PA in terms of Turing's degrees of computability. Computability is strongly connected to completeness. Actually, we should say that incompleteness is a subclass of incomputability. To compute a function, we need the notion of algorithm which is a set of finite informal instructions. If we want to compute a function, we have to follow all informal steps of an algorithm. However, we have always to cope with informal instructions. Alan Turing and Kurt Gödel were focusing at rendering the informal notion of algorithm formal. Gödel's recursive functions emerge from the logic, and so are very useful for formalizing algorithms. The definition of recursive functions is what we call an inductive definition. We start by defining a small class of very simple functions, called initial functions, to be recursive (base of induction). And then we introduce a small number of rules for deriving new recursive functions from those already obtained via the inductive process. We start with *primitive recursive functions*:

Definition 2 *(The primitive recursive functions)*
1) The initial functions (a)-(c) are primitive recursive:
(a) The Zero function defined by $0\,(n) = 0,\ \forall n \in \mathbb{N}$
(b) The successor function defined by $n' = n + 1,\ \forall n \in \mathbb{N}$
(c) The projection function U^{x}_{i} *defined by* $U^{x}_{i}(\vec{m}) = m_i$ *each* $x \geq 1$*, and* $i = 1, \ldots, x$

2) if g $h, h_0, \ldots, h_1$ are primitive recursive, then so f obtained obtained from g $h, h_0, \ldots, h_1$ by one of the rules:
(d) Substitution given by: $f(\vec{m}) = g(h_0(\vec{m}) \ldots h_1(\vec{m}))$
(e) Primitive recursion given by: $f(\vec{m}, 0) = g(\vec{m})$,
$f(\vec{m}, n+1) = h(\vec{m}, n, f(\vec{m}, n))$

The primitive recursive scheme describes how we inductively define value of f, getting $f(\vec{m}, n+1)$ via known primitive recursive functions in terms of the given parameters $\vec{m}$, the argument n, and the previously computed value $f(\vec{m}, n)$. Addition is primitive recursive since we have:

m + 0= m

m + (n + 1) = (m + n) + 1 = $(m+n)'$. Formally we have:

$f(m, 0) = U^1_1 = n$,

$f(m, n+1) = f(m, n)' = (U^3_3(m, n, f(m, n)))'$.

Multiplication is primitive recursive. In fact we have:

$m \times 0 = 0$

$m \times (n+1) = (m \times n) + m$. Predecessor function, recursive difference, absolute difference, remainder function, bounded sums and bounded product are all primitive recursive[4]. We could expect that with primitive recursive functions we have all computable functions. However by adopting nested recursion, in 1928 Ackermann defined a computable function A[5] which is not primitive recursive. Here we have the function:

A(m, 0) = m + 1

A(0, n + 1)= A(1, n)

A(m + 1, n + 1)= A (A(m, m +1), n).

Now we can introduce partial recursive functions:

Definition 3 *We say that a function $A :\longrightarrow B$ is total if $f(x)$ is defined for every $x \in A$, $f(x)\downarrow$.*

Otherwise if $f(x)$ is undefined ($f(x)\uparrow$) for some $x \in A$ we say that f is partial.

Now we define a larger class of functions, called partial recursive functions. This is done by the the following rule:

Definition 4 *(μ-operator or minimalization) If $g(\vec{n}, m)$ is partial recursive then so is f given by:*

$$f(\vec{n}) = \mu m[g(\vec{n}, m) = 0]$$

4. Fibonacci sequence is primitive recursive

5. Ackermann is not primitive recursive since it dominates all primitive recursive functions as the Busy Beaver Function dominates all URM program functions

where $\mu m[g(\vec{n}, m) = 0] = m_0 \longleftrightarrow g(\vec{n}, m_0) = 0$ *and* $(\forall m < m_0)[g(\vec{n}, m) \downarrow\neq 0]$

The μ operator is a search operation. It says compute $g(\vec{n}, 0) \ldots$ etc until we have $g(\vec{n}, m_0) = 0$. So m_0 is the value wanted. Surely, this search might go for ever if no such m_0 exists, in which case $f(\vec{n})$ does not get defined. Thus we have this large class of partial recursive functions. These functions can be defined from initial functions, using finite applications of Substitution and primitive recursive scheme, and finally from the μ-operator.

At this point, let's introduce Church's thesis.

Definition 5 *(Church) f is effectively computable if and only if it is partial recursive.*

Thanks to this thesis, the informal side of computation (algorithm) is combined with the formal side of computation (partial recursive functions). f is effectively computable if there exists some description of an algorithm, in some language, which can be used to compute any value $f(x)$ for which $f(x)\downarrow$. Church's thesis is independent from the language for computing. We establish a strong equivalence between all models of computations and formulate Church's thesis for all these different models (Lambda calculus, Turing machine, and unlimited register machine). Functions, that can be computed, are the same independently of the model of computation that we adopt. Church's thesis states that if someone can give a description of an algorithm for computing f, then there is a description of f as a partial recursive function or a Turing machine or in Lambda calculus or as an unlimited register machine. Church's thesis is true until now, because nobody has been able to find a counterexample to this thesis. However, it is possible to conceive a counterfactual situation or, possible world, where someone is capable of constructing an algorithm for computing $f(x)$ which does not have a formal description as a partial recursive function or as a Turing machine. By considering Church's thesis as true, we are introducing a temporal component in our world of mathematics. Church's thesis is true until now, but we cannot exclude that in the future someone will disprove it (finding a particular informal algorithm). Furthermore, we can say that Church's thesis is potentially true and has a temporal component (I will clarify these notions immediately after the introduction of some ideas related to intuitionism). When someone proves a theorem, according to classical mathematics, this theorem is atemporally true and actual true (I will explain this notion immediately). In classical mathematics, a truth does not have the dimension of time and is atemporal, because a proposition is true also before that a proof is constructed. Truths are outside the dimension of time and by constructing proofs, according to the classical vision of mathematics, we simply discover and capture them. In the case of Church's thesis, there is a temporal component, namely until now it is true. Church's thesis has a temporal component. Maybe, we should adopt a different conception of mathematics, such as intuitionism where the notion of time comes into the realm of mathematics. As Church's thesis, also the consistency of ZFC has a temporal component. Because of Gödel's second incompleteness theorem, we cannot prove directly the consistency of ZFC. Of course, we can trust the ZFC system, but we cannot exclude that in the future someone will discover a contradiction in it. Thus, ZFC is consistent until now. It has a temporal component. For the consistency of ZFC as for the truth of Church thesis, there is a temporal component which forces us to consider intutionism. To clarify this conception, I want to discuss some ideas related to intuitionism. Brouwer, the father of intuitionism, considered mathematics as activity of mental construction independent from the language. So, for Brouwer, Logic was not essential to mathematics. For Brouwer, a mathematical proposition is true when we can show a construction of it. At the beginning of his thought, Brouwer was rejecting hypothetical constructions and contradictions, but then he adopted the same view of Heyting, the other father of mathematical intuitionism. According to Heyting, $\neg A$ is true if the hypothesis that A is true causes a contradiction. This is the *hypothetical interpretation* of negation which characterizes the conception of Heyting. In 1923, Brouwer accepted hypothetical constructions and contradictions. In fact, he took position against mathematics without negation conceived by Griss. While for Brouwer mathematics was an activity without need of any languages, for Heyting language was essential for mathematics in order to communicate mathematical constructions. In fact, Heyting developed intuitionistic logic because he was thinking to render mathematics communicable in a formal language. According to Heyting, the fundamental activity of our mind is that of creating entities. This construction of abstract entities is the foundation of intuitionistic mathematics. Heyting rejects a platonistic-realistic philosophy of mathematics. In fact, in 1939, he wrote:

> An intuitionistic mathematician would not take position against a philosophy which holds that mind, during his creative activity, reproduces entities of a transcendent world, but he would consider this doctrine too speculative as foundation of pure mathematics. [Heyting 1939]

Heyting rejects the idea that there is a transcendent world of mathematics independent from human mind, which renders mathematical propositions true or false, but for Heyting mathematics is a **creation** of human mind. Furthermore, he wants to change the classical vision of mathematics by saying that truth is not anymore the fundamental notion but intuitionistic