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PREFACE

The papers injcluded in this volume contain the contributions given by several participants to the International Conference on Topological Algebras and Applications (ICTAA2022) organized on line by the Department of Mathematics and Computer Science of Palermo University (Italy) in the period August 31 - September 2, 2022.
The conference was expected to be attended in presence but the aftermath of the COVID19 pandemic and the general political situation suggested converting this event to an online meeting.
ICTAA2022 has been the thirteenth of a series begun in 1999 in Tartu (Estonia). The subjects covered in the conference were the traditional ones: Categories of Topological Algebras, Topological Rings, Topological Linear Spaces, Topological Modules, Topological Groups and Semigroups, Bornological Structures, Sheaf Theory, Bundle Theory, Topological K-theory, Operator algebras etc.
Thirty mathematicians, from thirteen countries, participated to the meeting and almost all presented the results of their recent research during the meeting.
We thank all participants and speakers for their cooperation in making of ICTAA2022 a successful event from the scientific point of view.
Unfortunately, all the aspects of sociability that usually accompany a conference were missing, but this did not depend on the will of the organizers or on that one of the participants.
We hope that the next ICTAA will be a real occasion of meeting personally old and new colleagues and friends working in the field of Topological Algebras and Applications.
A special thank is due to Dr. Giuseppe Russo for his technical help which made possible the organization of ICTAA2022 on-line.

The Organizers

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The property of being a Segal topological algebra is not always transitive

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Abstract

In [1] the question "Is the property of being a Segal topological algebra always transitive?" remained unanswered. In this paper we give the examples of left, right and two-sided Segal topological algebras for which the transitivity property does not hold, answering the question negatively.

Key words: Topological algebras; Segal topological algebras; matrix algebras; left ideals; right ideals; two-sided ideals
MSC: Primary 46H05; Secondary 16D25, 46H10

Contents

1 Introduction

All algebras in this paper are associative algebras over the field $\mathbb{K}$, which may stand for either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. There is no assumption of the existence of the unit in our algebras.

By a **topological algebra** we mean a topological vector space over $\mathbb{K}$ in which there is defined a separately continuous multiplication, making the vector space an algebra over $\mathbb{K}$.

Let us recall that a topological algebra (A, τ_A) is a left (right or two-sided) **Segal topological algebra** in a topological algebra (B, τ_B) via an algebra homomorphism $f : A \to B$, if

1) $\mathrm{cl}_B(f(A)) = B$, i.e., $f(A)$ is dense in B;

2) $\tau_A \supseteq \{f^{-1}(U) : U \in \tau_B\}$, i.e., f is continuous;

3) $f(A)$ is a left (respectively, right or two-sided) ideal of B.

In what follows, a left (right or two-sided) Segal topological algebra will be denoted shortly by a triple (A, f, B).

In this paper we give examples of the pairs (A, f, B) and (B, g, C) of left (right or two-sided) Segal topological algebras such that $(A, g \circ f, C)$ is not a Segal topological algebra. For that, we use some subalgebras of the algebra $M_3(\mathbb{K})$ of 3 by 3 matrices with all elements from the field $\mathbb{K}$.

2 Examples

Consider the following subsets of the algebra $M_3(\mathbb{K})$:

$$C = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & g \end{pmatrix} : a, b, c, d, e, f, g \in \mathbb{K} \right\},$$

$$B_l = \left\{ \begin{pmatrix} 0 & b & c \\ 0 & e & f \\ 0 & 0 & g \end{pmatrix} : b, c, e, f, g \in \mathbb{K} \right\},$$

$$B_r = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ d & e & f \\ 0 & 0 & g \end{pmatrix} : d, e, f, g \in \mathbb{K} \right\},$$

$$B_t = \left\{ \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & f \\ 0 & 0 & 0 \end{pmatrix} : c, f \in \mathbb{K} \right\},$$

$$A = \left\{ \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : c \in \mathbb{K} \right\},$$

$$A_r = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ d & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : d \in \mathbb{K} \right\}.$$

Then it is easy to check, that A, A_r, B_l, B_r, B_t and C are all algebras over $\mathbb{K}$ with respect to the usual matrix addition and matrix multiplication.

Moreover, B_l is a left ideal of C, B_r is a right ideal of C and B_t is a two-sided ideal of C, A is a left ideal of B_l and a two-sided ideal of B_t, A_r is a right ideal of B_r.

Take

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in C$$

and

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in A, \quad R = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in A_r.$$

Then

$$MP = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \notin A, \quad RN = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \notin A_r.$$

This means that A is not a left ideal of C (hence, A is also not a two-sided ideal of C) and A_r is not a right ideal of C.

Equip C with the trivial topology $\tau_C = \{\emptyset, C\}$ and B_l, B_r, B_t, A, A_r with the restrictions of τ_C to the respective subsets of C. Then the topologies $\tau_{B_l}, \tau_{B_r}, \tau_{B_t}, \tau_A$ and τ_{A_r} are also trivial topologies on respective algebras (hence, all algebraic operations in these algebras are separately and jointly continuous) and we obtain topological algebras $(A, \tau_A), (A_r, \tau_{A_r}), (B_l, \tau_{B_l}), (B_r, \tau_{B_r}), (B_t, \tau_{B_t})$ and (C, τ_C). As the topologies on these topological algebras are trivial, then all nonempty subsets of these algebras are dense and all maps between these algebras are continuous (in the trivial topology, there are no more open sets than empty set and the whole algebra itself, hence there are no more closed subsets than the empty set and the whole algebra itself).

Let $1_C : C \to C$ be the identity map, i.e., $1_C(M) = M$ for every $M \in C$. Set $1_{B_l} = 1_C|_{B_l} : B_l \to C, 1_{B_r} = 1_C|_{B_r} : B_r \to C, 1_{B_t} = 1_C|_{B_t} : B_t \to C$, $1_l = 1_{B_l}|_A : A \to B_l, 1_r = 1_{B_r}|_{A_r} : A_r \to B_r, 1_t = 1_{B_t}|_A : A \to B_t$, $1_A = 1_C|_A : A \to C$ and $1_{A_r} = 1_C|_{A_r} : A_r \to C$. Then $1_A = 1_{B_l} \circ 1_l = 1_{B_t} \circ 1_t$ and $1_{A_r} = 1_{B_r} \circ 1_r$. Moreover, all the maps $1_A, 1_{A_r}, 1_{B_l}, 1_{B_r}, 1_{B_t}, 1_l, 1_r, 1_t$ and 1_C are continuous algebra homomorphisms.

Now it is evident that $(A, 1_l, B_l), (B_l, 1_{B_l}, C)$ are left Segal topological algebras but $(A, 1_{B_l} \circ 1_l, C) = (A, 1_A, C)$ is not a left Segal topological algebra, because A is not a left ideal of C.

Similarly $(A_r, 1_r, B_r), (B_r, 1_{B_r}, C)$ are right Segal topological algebras but $(A_r, 1_{B_r} \circ 1_r, C) = (A_r, 1_{A_r}, C)$ is not a right Segal topological algebra, because A_r is not a right ideal of C.

Moreover, $(A, 1_t, B_t), (B_t, 1_{B_t}, C)$ are two-sided Segal topological algebras but $(A, 1_{B_t} \circ 1_t, C) = (A, 1_A, C)$ is not a two-sided Segal topological algebra because A is not a two-sided ideal of C.

3 Conclusion

We have found examples of pairs $(A, f, B), (B, g, C)$ of Segal topological algebras in the left hand case, right hand case and two-sided case such that $(A, g \circ f, C)$ is not a Segal topological algebra of the respective side. Hence, we can formulate the following theorem without any further proof.

Theorem 1. *None of the properties of being either a left Segal topological algebra, a right Segal topological algebra or a two-sided Segal topological algebra is transitive, in general.*

Although these properties are not transitive, in general, there are several classes of topological algebras for which the transitivity holds. For descriptions of the properties on Segal topological algebras, that are sufficient for the transitivity of the property of being a left (right or two-sided) Segal topological algebra, see [1], Theorem 1, Corollaries 3–7, pp. 158–159.

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References

[1] M. Abel, About the transitivity of the property of being Segal topological algebra, Acta Comment. Univ. Tartu. Math. **26** (2022), no. 1, 153–160.

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Waelbroeck algebras

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Abstract

Several structural properties of Waelbroeck algebras over $\mathbb{K}$ (one of the fields $\mathbb{R}$ of real numbers or $\mathbb{C}$ of complex numbers) are given. One-sided Waelbroeck algebras over $\mathbb{K}$ are introduced and studied. Three open problems are presented.

Key words: Topological algebra, Waelbroeck algebra, one-sided Waelbroeck algebra, von Neumann bornology, idempotently pseudoconvex bornology, radius of boundedness.
MSC: Primary 46H05; Secondary 46H20

Contents

1 Preface

One of the most important topological algebras over $\mathbb{K}$, which has many similar properties of Banach algebra is a Waelbroeck algebra.

The concept of a Waelbroeck algebra for locally convex unital algebras was introduced by L. Waelbroeck in [9] under the name of „continuous inverse algebra". The name „Waelbroeck algebra" was first used by R. Ouzilou for locally convex unital algebras in [8] and by A. Mallios for arbitrary topological unital algebras in [6]. Main structural properties of (not necessarily locally convex) Waelbroeck algebras are given in the present paper and one-sided Waelbroeck algebras over $\mathbb{K}$ are introduced and studied.

2 Introduction

Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ of real numbers or $\mathbb{C}$ of complex numbers, A an algebra over $\mathbb{K}$, Inv$A$ the collection of all invertible elements of A and Qinv$A$ the collection of all quasi-invertible elements of A ($a \in A$ is quasi-invertble in A, if there exists an element $a^q \in A$ such that $a \circ a^q = a + a^q - aa^q = \theta_A$ (the zero element of A) and $a^q \circ a = a^q + a - a^q a = \theta_A$).

1. A topological algebra (A, τ) over $\mathbb{K}$ is called a *Q-algebra*, if Qinv$A \in \tau$. A Q-algebra over $\mathbb{K}$, in which the quasi-inversion $\epsilon^q : a \to a^q$ is continuous, is called a *Waelbroeck algebra*. In the particular case, when A is a unital algebra, (A, τ) is called a Q-algebra, if Inv$A \in \tau$, and a Waelbroeck algebra, if Inv$A \in \tau$ and the inversion $\epsilon^{-1} : a \to a^{-1}$ in (A, τ) is continuous. For example, every Banach algebra over $\mathbb{K}$; every complete k-normed algebra over $\mathbb{K}$ with $0 < k \leqslant 1$; every F-algebra over $\mathbb{K}$ (that is, a topological algebra over $\mathbb{K}$, whose topology is defined by a submultiplicative F-norm) and several other well-known topological algebras are Waelbroeck algebras.

It is known (by Proposition 16 in [10]) that the multiplication is jointly continuous in every commutative Waelbroeck algebra over $\mathbb{K}$.

2. Let (A, τ) be a topological algebra over $\mathbb{C}$. For a given $a \in A$, let $\mathrm{sp}_A(a)$ be the spectrum of a, $\rho_A(a)$ the spectral radius of a,

$$S(a, \lambda) = \Big\{ \Big(\frac{a}{\lambda}\Big)^n : n \in \mathbb{N} \Big\}$$

for each $\lambda \neq 0$ and let $\beta_A(a)$ be the radius of boundedness of a, that is,

$$\beta_A(a) = \inf\{\lambda > 0 : S(a, \lambda) \text{ is bounded in } A\}$$

and $\inf(\emptyset) = \infty$. An element $a \in A$ is *bounded*, if $\beta_A(a) < \infty$.

3. Let (A, τ) be a topological algebra over $\mathbb{K}$, $\mathcal{B}_A$ the von Neumann bornology on (A, τ) (that is, the set of all bounded sets in (A, τ)),

$$\Gamma_k(U) = \Big\{ \sum_{v=1}^{n} \lambda_v u_v : n \in \mathbb{N}, u_1, \dots, u_n \in U, \lambda_1, \dots, \lambda_n \in \mathbb{K} \text{ with } \sum_{v=1}^{n} |\lambda_v|^k \leqslant 1 \Big\}$$

for each $k \in (0, 1]$ and $U \subset A$. The von Neumann bornology $\mathcal{B}_A$ is called

a) *convex*, if $\Gamma_1(U)$ is bounded in (A, τ) for each bounded subset U of (A, τ);

b) *pseudoconvex*, if there exists $k \in (0, 1]$ such that $\Gamma_k(U)$ is bounded in (A, τ) for each bounded subset U of (A, τ)
and

c) *idempotently pseudoconvex*, if there exists $k \in (0,1]$ such that $\Gamma_k(U)$ is bounded in (A,τ) for each idempotent and bounded subset U of (A,τ).

Hence, the convex von Neumann bornology is pseudoconvex and the pseudoconvex von Neumann bornology is idempotently pseudoconvex.

3 Main structural properties

First we give necessary and sufficient conditions for a topological algebra to be a Waelbroeck algebra.

Theeorem 1. a) *A topological algebra (A,τ) over $\mathbb{K}$ is a Waelbroeck algebra if and only if*

1) *there exists a neighbourhood O of zero in (A,τ) such that $O \subset \mathrm{Qinv}A$*
and
2) *the quasi-inversion $\epsilon^q : a \to a^q$ in (A,τ) is continuous at θ_A.*

b) *A unital topological algebra (A,τ) over $\mathbb{K}$ is a Waelbroeck algebra if and only if*

1') *there exists a neighbourhood O of e_A in (A,τ) such that $O \subset \mathrm{Inv}A$*
and
2') *the inversion $\epsilon^{-1} : a \to a^{-1}$ in (A,τ) is continuous at e_A.*

Proof. Let (A,τ) be a Waelbroeck algebra over $\mathbb{K}$. Then (A,τ) has properties 1) and 2) and, in the unital case, the properties 1') and 2').

a) Let (A,τ) be a topological algebra over $\mathbb{K}$, which has the property 1), $O \subset \mathrm{Qinv}A$ be a neighbourhood of zero in (A,τ) (defined by the condition 1)), f_a the map, defined by $f_a(b) = a + b$ for each $a, b \in A$, h_a the map, defined by $h_a(b) = ba$ for each $a, b \in A$, i the identity map on A, F_a the map, defined by $F_a(b) = b \circ a$ for each $a, b \in A$, and let $a_0 \in \mathrm{Qinv}A$. Then $F_{a_0^q}(a_0) = \theta_A$. Because $F_{a_0^q} = f_{a_0^q} \circ (i + h_{-a_0^q})$ is continuous as a composition of continuous maps, then there exists a neighbourhood $O(a_0)$ of a_0 in (A,τ) such that $F_{a_0^q}(O(a_0)) \subset O$. Thus, $b \circ a_0^q \in \mathrm{Qinv}A$ for each $b \in O(a_0)$. Since

$$b = b \circ \theta_A = b \circ (a_0^q \circ a_0) = (b \circ a_0^q) \circ a_0$$

for each $b \in O(a_0)$, then

$$b^q = ((b \circ a_0^q) \circ a_0)^q = a_0^q \circ (b \circ a_0^q)^q \in A$$

(because $(a \circ b)^q = b^q \circ a^q$ for each $a, b \in \mathrm{Qinv}A$) for each $b \in O(a_0)$. Hence $O(a_0) \subset \mathrm{Qinv}A$. Therefore, $\mathrm{Qinv}A \in \tau$.

Let now (A,τ) has the property 2) and let $a_0 \in \mathrm{Qinv}A$. Then

$$\lim_{b \to \theta_A} \epsilon^q(b) = \epsilon^q(\theta_A) = \theta_A,$$

Because the map $f : A \to A$, defined by $f(a) = a - a_0^q a$ for each $a \in A$, is continuous and $f(\theta_A) = \theta_A$, then there exists a neighbourhood U of zero in (A,τ) such that

$f(U) \subset \mathrm{Qinv}A$. Therefore $(u - a_0^q u)^q$ exists for each $u \in U$. Let $O(a_0) = a_0 + U$ and $a \in O(a_0)$. Then $a = a_0 + u$ for some $u \in U$. Because $(a_0 + u)^q = (u - a_0^q u)^q \circ a_0^q$ (see [5], Proposition 1.1.31), then

$$\lim_{a \to a_0} \epsilon^q(a) = \lim_{u \to \theta_A} (a_0 + u)^q = (\lim_{u \to \theta_A} (u - a_0^q u)^q) \circ a_0^q = \theta_A \circ a_0^q = a_0^q.$$

It means that ϵ^q is continuous at each $a_0 \in \mathrm{Qinv}A$. Consequently, (A, τ) is a Waelbroeck algebra over $\mathbb{K}$.

b) Let (A, τ) be a unital topological algebra over $\mathbb{K}$ which have the properties (1') and (2') and let $a_0 \in \mathrm{Inv}A$. Then, there is a balanced neighbourhood $O(e_A) \subset \mathrm{Inv}A$ of e_A. Since $O(e_A) = e_A + O$ for some balanced neighbourhood O of zero and

$$e_A - a \in e_A + (-O) \subset e_A + O \subset \mathrm{Inv}A$$

for each $a \in O$, then

$$a^q = e_A - (e_A - a)^{-1}$$

exists in A. Hence, $O \subset \mathrm{Qinv}A$ and

$$\lim_{a \to \theta_A} \epsilon^q(a) = e_A - \lim_{a \to \theta_A} \epsilon^{-1}(e_A - a) = \theta_A.$$

So, by the part a) of this proof, $\mathrm{Qinv}A \in \tau$ and the quasi-inversion in (A, τ) is continuous at every $a \in \mathrm{Qinv}A$. Therefore $\mathrm{Inv}A \in \tau$, because $f_{e_A} \circ g_{-1}$ is a homeomorphism from $\mathrm{Qinv}A$ onto $\mathrm{Inv}A$ (here g_{-1} is the map, defined by $g_{-1}(a) = -a$ for each $a \in A$) and $\mathrm{Inv}A = e_A - \mathrm{Qinv}A = (f_{e_A} \circ g_{-1})(\mathrm{Qinv}A)$. Since, $e_A - a \in \mathrm{Qinv}A$ for each $a \in \mathrm{Inv}A$, then

$$\lim_{a \to a_0} \epsilon^{-1}(a) = e_A - \lim_{a \to a_0} \epsilon^q(e_A - a) = e_A - \epsilon^q(e_A - a_0) = \epsilon^{-1}(a_0)$$

(because $a^{-1} = e_A - (e_A - a)^q$ for each $a \in \mathrm{Inv}A$). Hence, the inversion ϵ^{-1} is continuous at a_0. Consequently, (A, τ) is a unital Waelbroeck algebra over $\mathbb{K}$. □

The next result gives main structural properties of Waelbroeck algebras.

Theorem 2. *The following statements are true:*

a) *if (A, τ) is a Waelbroeck algebra over $\mathbb{K}$, (B, τ') a topological algebra over $\mathbb{K}$ and $\varphi : A \to B$ a continuous and open surjective homomorphism, then (B, τ') is a Waelbroeck algebra over $\mathbb{K}$;*

b) *if (A, τ) is a Waelbroeck algebra over $\mathbb{K}$ and I a two-sided ideal in A, then the quotient algebra A/I in the quotient topology is a Waelbroeck algebra over $\mathbb{K}$;*

c) *if A is a finite direct product of Waelbroeck algebras (A_i, τ_i) over $\mathbb{K}$, then A is a Waelbroeck algebra over $\mathbb{K}$ in the product topology;*

d) *(A, τ) is a Waelbroeck algebra over $\mathbb{K}$ if and only if the unitization $A_{\mathbb{K}}$ of A is a Waelbroeck algebra over $\mathbb{K}$ in the product topology;*

e) *if* (A, τ) *is a Hausdorff Waelbroeck algebra over* $\mathbb{K}$, B *is a closed subalgebra of* (A, τ) *and* $\beta_A(a) \leqslant \rho_A(a)$ *for each* $a \in A$, *then (in the subspace topology)* B *is a Waelbroeck algebra over* $\mathbb{K}$
and

f) *if* (A, τ) *is a unital Waelbroeck algebra over* $\mathbb{C}$ *for which the topological dual space of* A *has a non-zero element and, for any* $a \in A$, *there exists a neighbourhood* O *of zero in* $\mathbb{C}$ *such that for some* $\lambda \in O$ *from the weak boundedness of the set* $\{(\lambda a)^n : n \in \mathbb{N}\}$ *follows the boundedness of this set in* (A, τ), *then all elements of* A *are bounded.*

Proof. a) Let (A, τ) be a Waelbroeck algebra over $\mathbb{K}$, (B, τ') a topological algebra over $\mathbb{K}$ and $\varphi : A \to B$ a continuous and open surjective homomorphism. Then $\varphi(\text{Qinv}A) \subset \text{Qinv}B$ because $\varphi(a)^q = \varphi(a^q)$ for each $a \in \text{Qinv}A$, φ is a homomorphism and $\varphi(\theta_A) = \theta_B$. Since $\text{Qinv}A$ is an open set in (A, τ) and φ is an open surjective map, then $\varphi(\text{Qinv}A)$ is an open set in (B, τ'). Hence, $\varphi(\text{Qinv}A)$ is a neighbourhood of θ_B in (B, τ). So, there exists a neighbourhood O of zero in (B, τ) such that $O \subset \text{Qinv}B$.

To show that the quasi-inversion ϵ^q in (B, τ') is continuous at θ_B, let O be an arbitrary neighbourhood of zero in $\text{Qinv}B$. Then there is a neighbourhood O_B of zero in (B, τ') such that $O = O_B \cap \text{Qinv}B$ in the subset topology. Because φ is continuous, then there exists a neighbourhood O_A of zero in (A, τ) such that $\varphi(O_A) \subset O_B$. Hence, $O_A \cap \text{Qinv}A$ is a neighbourhood of zero in $\text{Qinv}A$ in the subset topology. Since the quasi-inversion ϵ^q is continuous at θ_A, there exists an open neighbourhood O'_A of zero in (A, τ) such that

$$\epsilon^q(O'_A \cap \text{Qinv}A) \subset O_A \cap \text{Qinv}A.$$

Let $O'_B = \varphi(O'_A \cap \text{Qinv}A)$. Then O'_B is an open neighbourhood of zero in (B, τ') due to the openness of $\text{Qinv}A$ and the openness of φ. Because

$$O'_B \subset \varphi(O'_A) \cap \varphi(\text{Qinv}A) \subset \varphi(\text{Qinv}A) \subset \text{Qinv}B,$$

then O'_B is a neighbourhood of zero in $\text{Qinv}B$. Moreover,

$$\epsilon^q(O'_B) = \epsilon^q(\varphi(O'_A \cap \text{Qinv}A)) = \varphi(\epsilon^q(O'_A \cap \text{Qinv}A)) \subset \varphi(O_A \cap \text{Qinv}A) \subset O_B \cap \text{Qinv}B.$$

Therefore the quasi-inversion ϵ^q is continuous at θ_B. Consequently, (B, τ') is a Waelbroeck algebra over $\mathbb{K}$, by Theorem 1.

b) Let (A, τ) be a Waelbroeck algebra over $\mathbb{K}$, I a two-sided ideal in A and $\pi_I : A \to A/I$ the quotient map. Then A/I (in the quotient topology) is a topological algebra over $\mathbb{K}$ with respect to the usual algebraic operations in A/I. Since π_I is a continuous and open surjective homomorphism in the quotient topology on A/I, then A/I (in the quotient topology) is a Waelbroeck algebra over $\mathbb{K}$, by the statement a) above.

c) Let $n \in \mathbb{N}$, $(A_1, \tau_1), \ldots, (A_n, \tau_n)$ be Waelbroeck algebras over $\mathbb{K}$ and $A = \prod_{i=1}^n A_i$ the direct product of algebras A_i. If we define the algebraic operations in A point-wise, then A (in the product topology) is a topological algebra over $\mathbb{K}$. It is easy to see that

$$\mathrm{Qinv}A = \prod_{i=1}^{n} \mathrm{Qinv}A_i.$$

Because every (A_i, τ_i) is a Waelbroeck algebra, then there exists a neighbourhood O_i of θ_{A_i} in (A_i, τ_i) such that $O_i \subset \mathrm{Qinv}A_i$ for each $i \in \mathbb{N}_n$. Therefore,

$$\theta_A = (\theta_{A_1}, \ldots, \theta_{A_n}) \in O = \prod_{i=1}^{n} O_i \subset \prod_{i=1}^{n} \mathrm{Qinv}A_i = \mathrm{Qinv}A.$$

In the product topology, the set O is a neighbourhood of zero. So, we have shown that there exists a neighbourhood O of zero in A (in the product topology) such that $O \subset \mathrm{Qinv}A$.

To show that the quasi-inversion ϵ^q in A is continuous at θ_A in the product topology, let O be any neighbourhood of zero in A in the product topology. Then, there exist neighbourhoods O_i of zero in (A_i, τ_i) such that $\prod_{i=1}^{n} O_i \subset O$. Since the quasi-inversion ϵ_i^q in every (A_i, τ_i) is continuous at θ_{A_i}, then, for each i, there exists a neighbourhood U_i of zero in (A_i, τ_i) such that $\epsilon_i^q(U_i) \subset O_i$. Therefore,

$$\epsilon^q\Big(\prod_{i=1}^{n} U_i\Big) = \prod_{i=1}^{n} \epsilon_i^q(U_i) \subset \prod_{i=1}^{n} O_i \subset O.$$

Because $\prod_{i=1}^{n} U_i$ is a neighbourhood of zero in A in the product topology, then the quasi-inversion ϵ^q in A is continuous at θ_A. So, by Theorem 1, the finite direct product of Waelbroeck algebras over $\mathbb{K}$ is a Waelbroeck algebra over $\mathbb{K}$.

d) Let (A, τ) be a Waelbroeck algebra over $\mathbb{K}$. Then (A, τ) is a Q-algebra. Therefore, $A_{\mathbb{K}}$ is a Q-algebra in the product topology (see [1], Proposition 2). Since $A_{\mathbb{K}}$ (in the product topology) is a topological algebra over $\mathbb{K}$, then the map

$$F : A \times (\mathbb{K} \setminus \{0\}) \to A,$$

defined by $F((a, \lambda)) = \lambda a$ for each $(a, \lambda) \in A \times (\mathbb{K} \setminus \{0\})$, is jointly continuous and $F((\theta_A, 1)) = \theta_A$. Hence, there exists a neighbourhood O of $(\theta_A, 1)$ in $A_{\mathbb{K}}$ (in the product topology) such that $F(O) \subset \mathrm{Qinv}A$. Moreover, there exists an element $U \times V$ in a base of neighbourhoods of $(\theta_A, 1)$ in the product topology, where U is a neighbourhood of zero in (A, τ) (we can assume here that U is a balanced neighbourhood of zero, otherwise we take the balanced neighbourhood of zero $U' \subset U$ instead of U) and V a neighbourhood of 1 in $\mathbb{K}$ such that $U \times V \subset O$ and $UV \subset \mathrm{Qinv}A$. Let $W = \{\lambda \in \mathbb{K} : \frac{1}{\lambda} \in V\}$. Because the inversion in $\mathbb{K}$ is continuous, then W is a neighbourhood of 1 in $\mathbb{K}$ and $-\frac{1}{\lambda}a \in VU$ whenever $\lambda \in W$ and $a \in U$. The convergence in $A_{\mathbb{K}}$ in the product topology is the convergence by coordinates and

$$(a, \lambda)^{-1} = \Big(-\frac{1}{\lambda}\Big(-\frac{1}{\lambda}a\Big)^q, \frac{1}{\lambda}\Big)$$

for each $(\lambda, a) \in \text{Inv}A_{\mathbb{K}}$. Let Δ be the map, which is defined by $\Delta(a,\lambda) = (a,\lambda)^{-1}$ for each $(a,\lambda) \in \text{Inv}A_{\mathbb{K}}$. Since

$$\lim_{(a,\lambda)\to(\theta_A,1)} \Delta((a,\lambda)) = \Big(-\frac{1}{\lambda}\lim_{a\to\theta_A}\epsilon^q\Big(-\frac{1}{\lambda}a\Big), \lim_{\lambda\to 1}\frac{1}{\lambda}\Big) = \Big(-\frac{1}{\lambda}\theta_A, 1\Big) = (\theta_A, 1)$$

in $U \times W$, then the inversion in $A_{\mathbb{K}}$ (in the product topology) is continuous at $(\theta_A, 1)$. Hence, by the Theorem 1, $A_{\mathbb{K}}$ (in the product topology) is a unital Waelbroeck algebra over $\mathbb{K}$.

Let now $A_{\mathbb{K}}$ (in the product topology) be a Waelbroeck algebra over $\mathbb{K}$. Then $A_{\mathbb{K}}$ is a Q-algebra and the inversion in $A_{\mathbb{K}}$ is continuous at $(\theta_A, 1)$. Therefore, there exists a neighbourhood O of $(\theta_A, 1)$ such that $O \subset \text{Inv}A_{\mathbb{K}}$ and

$$\lim_{(a,\lambda)\to(\theta_A,1)} \Delta((a,\lambda)) = \Delta((\theta_A,1)) = (\theta_A,1).$$

Now (similarly as above), there exists a balanced neighbourhood U of zero in (A,τ) and a neighbourhood V of 1 in $\mathbb{K}$ such that such $U \times V \subset O$. Since $(-a,1) \in U \times V$ for each $a \in U$, then $(-a,1)^{-1}$ exists. Because $(-a,1)^{-1} = (-a^q,1) \in A_{\mathbb{K}}$, then $U \subset \text{Qinv}A$ and

$$\lim_{a\to\theta_A} a^q = -p(\lim_{(-a,1)\to(\theta_A,1)}(\Delta((-a,1))) = -p((\theta_A,1)) = \theta_A$$

(here p denotes the projection of $A_{\mathbb{K}}$ onto A, which is continuous). Hence, the quasi-inversion ϵ^q in (A,τ) is continuous at θ_A. So, (A,τ) satisfies the conditions 1') and 2') of Theorem 1, because of which (A,τ) is a Waelbroeck algebra over $\mathbb{K}$, by Theorem 1.

e) See the proof of Theorem 1 in [2].

f) See [3], pp. 63–64. □

Corollary 1. *Let (A,τ) be a Hausdorff Waelbroeck algebra over $\mathbb{C}$ with jointly continuous multiplication and idempotently pseudoconvex von Neumann bornology. If every element of* Qinv$A$ *is bounded, then every closed subalgebra of (A,τ) (in the subset topology) is a Waelbroeck algebra.*

Proof. By assumption, it is true that $\beta_A(a) \leqslant \rho_A(a)$ for each $a \in A$, by Proposition 4 in [2]. Therefore, every closed subalgebra B of (A,τ) (in the subset topology) is a Waelbroeck algebra, by the statement e) of Theorem 2. □

Corollary 2. *Let (A,τ) be a Hausdorff locally convex Waelbroeck algebra over $\mathbb{C}$ with jointly continuous multiplication. Then every closed subalgebra B of (A,τ) (in the subset topology) is a Waelbroeck algebra.*

Proof. See the proof of Corollary 4 in [2] (in the unital case, see the proof of Proposition 1.7 in [5]). □

Corollary 3. *Every element of any Hausdorff locally convex Waelbroeck algebra over $\mathbb{C}$ is bounded.*

Proof. For the unital case, see [3], p. 64. Let A be a non-unital algebra over $\mathbb{C}$ and $A_{\mathbb{C}}$ the unitization of A. Then $A_{\mathbb{C}}$ (in the product topology) is a unital Waelbroeck algebra over $\mathbb{C}$, by the statement d) of Theorem 2. Moreover, $A_{\mathbb{C}}$ is a Hausdorff locally convex space. Therefore, every element in $A_{\mathbb{C}}$ is bounded by the first part of the proof. Hence, there exists $\lambda_0 \in \mathbb{R}$ such that the set

$$V = \left\{ \left(\frac{(a,0)}{\lambda_0} \right)^n : n \in \mathbb{N} \right\}$$

is bounded in $A_{\mathbb{C}}$. Let O_A be any neighbourhood of zero in (A, τ) and O a neighbourhood of zero in $\mathbb{C}$. Then $O_A \times O$ is a neighbourhood of zero in $A_{\mathbb{C}}$ in the product topology. Thus, there is $M > 0$ such that $V \subset M(O_A \times O)$. Because

$$\left(\frac{(a,0)}{\lambda_0} \right)^n = \left(\left(\frac{a}{\lambda_0} \right)^n, 0 \right)$$

for each $n \in \mathbb{N}$, then

$$\left\{ \left(\frac{a}{\lambda_0} \right)^n : n \in \mathbb{N} \right\} \subset MO_A.$$

Hence, every element in A is bounded. □

4 One-sided Waelbroeck algebras

Let A be an algebra over $\mathbb{K}$, $\mathrm{Inv}_l A$ the collection of all left invertible elements in A and $\mathrm{Qinv}_l A$ the collection of all left quasi-invertible elements of A ($a \in A$ is left quasi-invertble in A, if there exists an element $a_l^q \in A$ such that $a_l^q \circ a = a_l^q + a - a_l^q a = \theta_A$). Similarly, let $\mathrm{Inv}_r A$ the collection of all right invertible elements in A and $\mathrm{Qinv}_r A$ the collection of all right quasi-invertible elements of A.

We say that a topological algebra (A, τ) over $\mathbb{K}$ is a *Q_l-algebra*, if $\mathrm{Qinv}_l \in \tau$, and a *left Waelbroeck algebra*, if (A, τ) is a Q_l-algebra, in which the left quasi-inversion $\epsilon_l^q : a \to a_l^q$ is continuous. In the particular case, when A is a unital algebra, we say that (A, τ) is a *Q_l-algebra*, if $\mathrm{Inv}_l A \in \tau$, and a left Waelbroeck algebra, if (A, τ) is a Q_l-algebra, in which the left inversion $\epsilon_l^{-1} : a \to a_l^{-1}$ is continuous. Similarly, we define a Q_r-algebra and a *right Waelbroeck algebra* over $\mathbb{K}$.

Theorem 3. a) *A topological algebra (A, τ) over $\mathbb{K}$ is a left (right) Waelbroeck algebra if and only if*

1) *there exists a neighbourhood O of zero in (A, τ) such that $O \subset \mathrm{Qinv}_l A$ (respectively, $O \subset \mathrm{Qinv}_r A$)*
and

2) *the left quasi-inversion* $\epsilon_l^q : a \to a_l^q$ (*respectively, the right quasi-inversion* $\epsilon_r^q : a \to a_r^q$) *in* (A,τ) *is continuous at* θ_A.

b) *A unital topological algebra* (A,τ) *over* $\mathbb{K}$ *is a left Waelbroeck algebra* (*right Waelbroeck algebra*) *if and only if*

1') *there exists a neighbourhood* O *of* e_A *in* (A,τ) *such that* $O \subset \mathrm{Inv}_l A$ (*respectively,* $O \subset \mathrm{Inv}_r A$)
and
2') *the left inversion* $\epsilon_l^{-1} : a \to a_l^{-1}$ (*respectively, the right inversion* $\epsilon_r^{-1} : a \to a_r^{-1}$) *in* (A,τ) *is continuous at* e_A.

Proof. The proof is similar to the proof of Theorem 1. □

We can also show that the analogue of Theorem 2 holds for one-sided Waelbroeck algebras.

Up to now, I don't know of any example of a left Waelbroeck algebra that is not a right Waelbroeck algebra. Therefore, I present the following open problems:

1. Does there exist a left Waelbroeck algebra over $\mathbb{K}$, which is not a right Waelbroeck algebras or vice versa?

2. Does there exist a topological algebra over $\mathbb{K}$, in which only one of the sets $\mathrm{Qinv}_l A$ or $\mathrm{Qinv}_r A$ (in the unital case, only one of the sets $\mathrm{Inv}_l A$ or $\mathrm{Inv}_r A$) is open?

3. Does there exist a topological algebra over $\mathbb{K}$, in which the left quasi-inversion (in the unital case, the left inversion) is continuous but the right quasi-inversion (respectively, the right inversion) is not continuous or vice versa.

Remark. In special cases, some of the results, presented here, are known: Theorem 1 for unital Waelbroeck algebras can be found in [6], Proposition 4.1, and partly in [7], p. 204; the statements a), b) and d) of Theorem 2 for Waelbroeck algebras over $\mathbb{K}$ with jointly continuous multiplication - in [4] (see Lemma 3.6.26, Corollary 3.6.27 and Proposition 3.6.28) and Theorem 3 for unital Waelbroeck algebras has been partly given (without proof) in [7], p. 205.

5 Conclusion

Several structural properties of Waelbroeck algebras over $\mathbb{K}$ are presented. One-sided Waelbroeck algebras over $\mathbb{K}$ are introduced and some properties of them are given. Three open problems are presented.

References

[1] M. Abel, *Dense subalgebras in noncommutative Jordan topological algebras*, Acta Comment Univ. Tartu Math. **1** (1996), 65–70.

[2] M. Abel, *When every closed subalgebra of a Waelbroeck algebra is a Waelbroeck algebra*. Proc. Intern. Conf. on Topol. Alg. and their Appl. – ICTAA 2018, 33−47, Math. Stud. (Tartu) **7**, (Est. Math. Soc., Tartu, 2018).

[3] M. Abel, *Topological algebras with all elements bounded.* Topol. Alg. and their Appl., ICTAA–2015, Holon, Israel, 2015. (Wilmington, DE: Academic Services. 59–71, 2020).

[4] V. K. Balachandran, *Topological algebras*. North-Holland Math. Studies **185**. (North-Holland Publishing Co., Amsterdam, 2000).

[5] H. Biller, *Continuous inverse algebras with involution.* Forum Math. 22 (2010) **6**, 1033−1059.

[6] A. Mallios, *Topological algebras. Selected topics.* North-Holland Mathematics Studies **124**. (North-Holland Publishing Co., Amsterdam, 1986).

[7] M. A. Naimark, *Normed rings* (Russian). (Izdat. „Nauka", Moscow, 1968).

[8] R. Ouzilou, *Algèbres involutives à adverse continu.* Publ. Dèp. Math. (Lyon) **2** (1965), 54−71.

[9] L. Waelbroeck, *Les algèbres à inverse continu.* C. R. Acad. Sci. Paris **238** (1954), 640−641.

[10] L. Waelbroeck, *Topological vector spaces and algebras.* Lecture Notes in Math. **230** (Springer-Verlag, Berlin-New York, 1971).

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Bounded elements in locally convex quasi *-algebras

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Abstract

We summarize in this note a series of basic concepts of the theory of locally convex quasi *-algebras and we propose some possible approach to the notion of *bounded element* which has revealed to be relevant as starting point of a spectral analysis.

Key words: quasi*-algebras, bounded elements
MSC: 46A03; 47L60

Contents

1 Introduction and Basic definitions

The theory of (locally convex) quasi *-algebras begun by Lassner [11, 12] in the 1980's, has reached a quite satisfactory status of richness and completeness: several aspects have been investigated and several applications have been considered. The aim of this note is to propose some ideas on a possible different approach to the notion of bounded element of a locally convex quasi *-algebra which requires the existence of