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DYNAMICS OF LINEAR OPERATORS

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To the whole Department of Mathematics and Physics of *Università degli Studi della Campania "L. Vanvitelli"*, I would like to express my thanks for providing me such a great and enjoyable opportunity of personal and professional growth.

Contents

Introduction

The *theory of dynamical systems* is the mathematical study of the long-term behavior of evolving systems, understood as maps acting on certain spaces. It originated at the end of the 19th century and, as is often the case in mathematics, its roots must be sought in the attempts to explain various concrete phenomena (like, citing one of the most famous, the evolution of the solar system), and to predict their behavior. Growing from a multitude of different roots, this theory has become over time a rich field interacting with numerous and different branches of mathematics and, in general, sciences. From the concrete context in which it developed, the theory of dynamical systems inherits one of its most important terms: the evolving states of a system, given by the iterations of a map on a certain space, are referred to as an *orbit* by analogy to the solar system. Properties of orbits like periodicity, chaotic behaviors, stability under perturbations, appear repeatedly in the study of dynamical systems, making the concept of orbit a fundamental of this theory. In its early stage, the theory of dynamical systems focused mainly on non-linear systems (i.e., systems generated by non-linear maps), considered more suitable, than the linear ones, to possess strange behaviors and therefore more intriguing from a dynamical point of view. A decisive step in the opposite direction was made in the first half of the 20th century, when studies showed that even linear systems can behave unpredictably, leading to the birth of the so called *linear dynamics*.

This thesis aims at studying the behavior of dynamical systems generated by linear operators. The fundamentals of linear dynamics are investigated first in a general context and, then, in the specific setting of a class of versatile linear operators that has had an explosion of interest in the last decades: the composition operators $T_f : \varphi \mapsto \varphi \circ f$. Precisely, the thesis is organized into five chapters, each of them preceded by a brief introduction on the covered subject.

Chapter 1 introduces linear dynamical systems and provides all the background necessary for a better understanding of the whole thesis. It is a walk through the most important notions of linear dynamics, among which hypercyclicity, topological mixing, expansivity, shadowing, hyperbolicity, generalized hyperbolicity and, in particular, chaos. In the last decades, the term chaos has been applied to a variety of linear systems that exhibit some type of strange or random behaviors. This variety of approaches precludes a unique definition of the word “chaos”, giving rise to a multitude of notions, such as Devaney chaos and Li-Yorke chaos, just to cite some of them, both analyzed in the chapter and, in general, in this thesis. Moreover, the chapter deals with a well-known equivalence relation among dynamical systems: the conjugacy, essential in order to classify the behavior of dynamical systems. This relation, together with the weaker semi-conjugacy, is presented in the last part of the chapter, where new results, about properties preserved by semi-conjugation, are showed.

In Chapter 2, the notions introduced in the first chapter are brought into the more specific context of composition dynamical systems, i.e., systems generated by composition operators acting on L^p spaces, $1 \leq p < \infty$. First, the motivations that inspired this research are introduced. While studying the behavior of a dynamical system, it may be convenient to look for a conjugacy or semi-conjugacy with a better-understood system: this explains why the chapter opens with weighted shifts, known to be a good model for understanding the dynamics of various operators, among which composition operators. Then, the general picture of composition operators and, therefore, composition dynamical systems, is analyzed and widened by adding some new characterizations, like that of expansivity, to the already known ones. It turns out that hypercyclicity, topological mixing, Li-Yorke chaos and expansivity are completely understood in this context.

Unfortunately, the same cannot be said about chaos, frequent hypercyclicity, shadowing and generalized hyperbolicity: they are not easy to understand in the general context of composition dynamical systems. Although no characterization of these properties is known in such context, recently, it has been proved that chaos and frequent hypercyclicity coincide for a large class of these systems: the dissipative composition dynamical systems of bounded distortion. This motivates the contents of Chapter 3, which opens with a detailed description of the notions of dissipativity and bounded distortion. Then, among other results, necessary and sufficient conditions to get the shadowing property and generalized hyperbolicity are investigated in the chapter. These new conditions show that the two concepts are more than connected: they even coincide in the context of dissipative composition dynamical systems of bounded distortion. The analysis of the chapter is concluded with the presentation of elegant and useful computational tools: using them, it is shown how some natural probability distributions, such as the Laplace distribution and the Cauchy distribution, lead to composition operators, of dissipative systems of bounded distortion, with and without the shadowing property.

At this point of the thesis, after a careful look at the results of chapters 2 and 3, it should have become clear to the reader the intimate relation between weighted shifts and composition operators of dissipative systems of bounded distortion. Chapter 4 deals with a detailed description of this connection: after a brief introduction on the motivations leading to this research, it is here proved a clou theorem developing a new general method which takes a known characterization of a linear dynamical property for weighted shifts and translates it into the setting of composition operators of dissipative systems of bounded distortion. Finally, several examples are treated in detail, in conclusion of the extensive analysis of the previous chapters about the dynamics of composition dynamical systems, both in the general and in the dissipative case.

The last chapter of the thesis, Chapter 5, differs slightly from the above topics. It gives a brief overview on basic properties of composition operators on various function spaces different from the L^p spaces considered so far. In addition to the linear composition operator, unlike the other chapters, a type of non-linear composition operator is also introduced and studied here. The chapter is mainly centred on the following problem: find necessary and sufficient conditions for a composition operator to map a function space into itself. In this direction, many already known results are recalled, and new ones are described, contributing to a theory which is still far from being complete. The theory does not seem to run parallel for the two types of composition operators and, even in the linear case, the previous problem turns out to be, sometimes, quite difficult.

From this outline, it is evident that the goal of the thesis is not only to show new results in the theory of dynamical systems (and, expecially, linear dynamics) but, also, to provide an accurate global view on the subject. For this reason, the theory illustrated in the entire thesis is enriched by graphs, diagrams, examples and tables. Precisely, the tables (in the second and third chapter) provide a scheme of all the known characterizations, of the above mentioned properties, for composition dynamical systems, both in the general case and in the dissipative case with bounded distortion. In this way, the reader has a general picture of what is known, and what is still open, in the literature on the topic.

Notation and Terminology

Some recurrent terminology and notation in the thesis:

- $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$;
- $\mathbb{D}$ and $\mathbb{T}$ denote, respectively, the open unit disk and the unit circle in the complex plane $\mathbb{C}$;
- given a Banach space X, $S_X = \{x \in X \,:\, \|x\| = 1\}$, i.e., S_X denotes the *unit sphere* of X, and $B_X(x,r)$ denotes the *open ball* of radius $r > 0$ centered at $x \in X$;
- if T is a bounded operator on X, then $\sigma(T), \sigma_p(T), \sigma_a(T)$ and $\sigma_r(T)$ denote, respectively, the *spectrum*, the *point spectrum*, the *approximate point spectrum* and the *residual spectrum* of T, and $r(T)$ denotes the *spectral radius* of T (recall the *spectral radius formula:* $r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$ [40]);
- *orbit* is used for the T-*orbit* of $x \in X$, given by $Orb(x,T) = \{T^n(x),\ n \geq 0\}$, when T is understood.

Other notations used in the thesis (for example, to shorten formulas in some results or to facilitate the fluidity of a paragraph) will be introduced directly in the chapters in which they appear.

CHAPTER 1

Linear dynamics

The restriction of interesting dynamical behaviors to non-linear operators is quite common inside and outside mathematics: typically, the analysis of particular non-linear operators can realistically describe the evolution of certain natural, social or economic phenomena. Nevertheless, in the last decades, this view turned out to be inappropriately limited: also linear operators can exhibit very complex dynamics like, for instance, linear chaos. The area of mathematics mainly implicated in the study of the behavior of iterates of linear operators is called *linear dynamics*. A systematic study of linear dynamics has probably begun in 1982 with the PhD thesis of Kitai [47], and it finds its basis in the papers [36, 38] by the mathematicians Gethner, Godefroy and Shapiro. Since then, an extensive literature has been produced (for instance [11, 16, 17, 20, 22, 25, 37] and their references) showing that even linear dynamics can exhibit the same complexity and charm as non-linear dynamics. Comprehensive collections of recent results on this topic are the monographs [10, 40].

This chapter introduces the fundamentals of linear dynamics and, as this branch of mathematics lies in the intersection between operator theory and dynamical systems, the notions are presented in the context of linear operators first and in that of dynamical systems then. The setting is that of Banach spaces, but most of the results showed in the chapter can be extended to more general spaces, like Fréchet spaces [40]. Hence, unless otherwise stated, throughout the chapter, X denotes a Banach space, and an *operator* T on X means a bounded linear operator $T : X \to X$. In addition, it should be noted that definitions are sometimes accompanied by figures, which must be considered as support for the understanding of a given phenomenon, and not as its precise graphic description.

1.1 Definitions and background results

For the sake of clarity and fluency of the thesis, in this section, only the notions and the results which are used in other chapters are introduced. For a more accurate analysis of each notion and for further results on the topics, please refer to the references gradually provided in the entire section.

1.1.1 Types of chaos

Definition 1.1.1. *The operator T is said to be*

- topologically transitive *if for any pair of non-empty open subsets U, V of X, there is $k \in \mathbb{N}$ such that $T^k(U) \cap V \neq \emptyset$;*
- topologically mixing *if for any pair of non-empty open subsets U, V of X, there is $k_0 \in \mathbb{N}$ such that $T^k(U) \cap V \neq \emptyset$, for all $k \geq k_0$.*

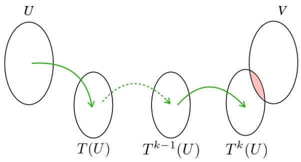

Figure 1.1: *Topological transitivity*

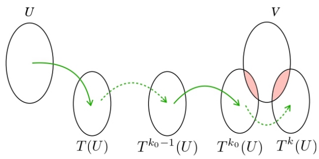

Figure 1.2: *Topological mixing*

Of course, topological mixing is a stronger form of topological transitivity, where topological transitivity simply means that the space X is irreducible under the action of T, i.e., any two non-trivial portions of X are always connected by T. Topological transitivity is, in linear dynamics, the first characteristic of an operator to be chaotic (according to Devaney): combining it with the properties to admit many points whose orbit has a regular behavior and to have a sensitive dependence on initial conditions, one can get the definition of Devaney chaos [30]. Such "sensitivity" on initial conditions means that small variations on the points of the space can bring large variations (and, therefore, unpredictability) on the relative orbits. Successively, it was showed by Banks et al. [6] that, in the Devaney's definition of chaos, sensitivity is superfluous as implicated by the other two ingredients.

Although the word "chaos" refers, in the literature, to Devaney's definition, there exist also other types of chaos, such as the one originally given by Li and Yorke in [52].

Definition 1.1.2. *The operator T is said to be*

- chaotic *if it is topologically transitive and has a dense set of periodic points;*
- Li-Yorke chaotic *if there is an uncountable set $U \subset X$, called* scrambled set*, such that for each $x, y \in U$, $x \neq y$,*

$$\liminf_{n\to\infty} \|T^n(x) - T^n(y)\| = 0 \;\; \textit{and} \;\; \limsup_{n\to\infty} \|T^n(x) - T^n(y)\| > 0.$$

The pair $\{x, y\}$ is called a Li-Yorke pair.

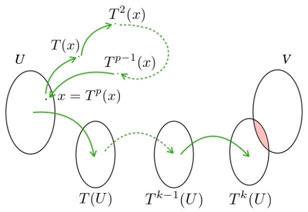

Figure 1.3: *Chaos*

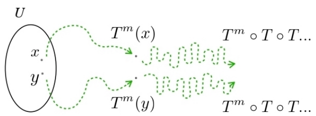

Figure 1.4: *Li-Yorke pair*

In the previous definition, the scrambled set can be thought of as a set where the orbits of any two distinct points become both arbitrarily close and sufficiently distant from each other. The following relation between Li-Yorke chaos, iterates of an operator and its spectrum, is investigated, among other results, in [14].

Corollary 1.1.3. [14, Corollary 6] *If T is Li-Yorke chaotic, then $\sigma(T) \cap \mathbb{T} \neq \emptyset$ and T^n is Li-Yorke chaotic, for each $n \in \mathbb{N}$.*

In the same article, it is showed that Li-Yorke chaos is equivalent to the property of T to admit an *irregular vector*, that is, a vector $x \in X$ such that

$$\varliminf_{n\to\infty} \|T^n(x)\| = 0 \quad \text{and} \quad \varlimsup_{n\to\infty} \|T^n(x)\| = \infty.$$

The origin of such vectors can be found in [13] and, surprisingly, it turns out that initially this concept of irregularity was not linked to Li-Yorke chaos. Since an irregular vector has an unbounded orbit and a subsequence converging to zero, the following property is showed in [16].

Proposition 1.1.4. [16, Propositions 3 and 5] *The following statements hold.*

(i) The set of all $x \in X$ such that $\{T^n(x)\}_{n\in\mathbb{N}}$ has a subsequence converging to zero is a G_δ set in X.

(ii) If T has a vector of unbounded orbit, then T has a residual set of vectors with unbounded orbits.

The previous proposition allows to highlight the already known connection between Li-Yorke chaos and irregular vectors, as the following result shows.

Proposition 1.1.5. *If T is Li-Yorke chaotic, then there exists a closed subset Y of X such that*

- *Y is T-invariant, that is $T(Y) \subseteq Y$;*
- *the restriction of T to Y, i.e. $T_{|Y}$, has a residual, and therefore dense, set of irregular vectors.*

Proof. As T is Li-Yorke chaotic, then it admits an irregular vector $x \in X$. Define

$$Y := \overline{span\{Orb(x,T)\}} = \overline{\left\{\sum_{i=1}^{k} a_i T^{n_i}(x) : a_i \in \mathbb{R}, n_i \in \mathbb{N}, k \in \mathbb{N}\right\}}.$$